Qualitative properties of solutions of the nonlinear Schrödinger equation on metric graphs

Graduate Lecture Series in Analysis and PDEs, Brown University

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Joint work with Colette De Coster (CERAMATHS/DMATHS, Valenciennes, France) and Christophe Troestler (UMONS, Mons, Belgium)

Thanks to Colette for the slides!

Tuesday 21 October 2025

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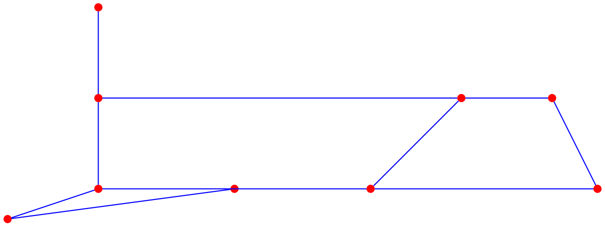
- you!

Compact metric graphs

A compact metric graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is a connected network made up of a finite number of finite length edges $e \in \mathbb{E}$, glued at vertices $\mathrm{v} \in \mathbb{V}$, according to the topology of a graph.



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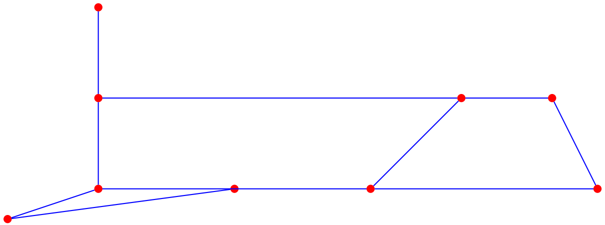

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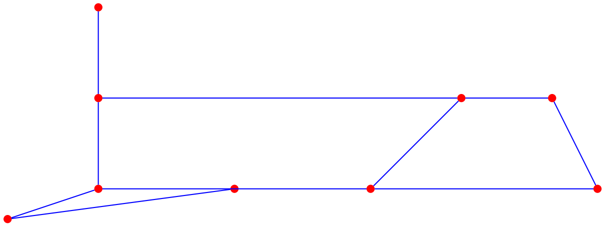


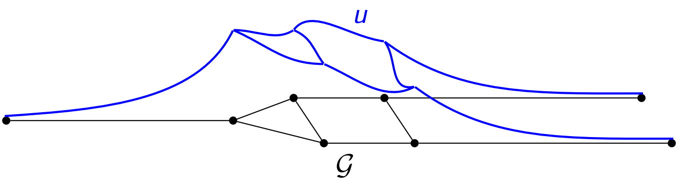
- Any bounded edge e is identified with a compact interval of $\mathbb{R}$;
- $u \in L^p(\mathcal{G}) \iff u \in L^p(e)$ for every edge e of $\mathcal{G}$.

The Sobolev space $H^1(\mathcal{G})$

The Sobolev space $H^1(\mathcal{G})$ is defined as follows

$$u \in H^1(\mathcal{G}) \iff \begin{cases} u \in H^1(e) & \text{for every edge } e \text{ of } \mathcal{G}, \\ u : \mathcal{G} \to \mathbb{R} & \text{is continuous on } \mathcal{G}. \end{cases}$$

Here is what a typical $H^1(\mathcal{G})$ function looks like:



The differential system

Given constants $p > 2$ and $\lambda > 0$, we are interested in solutions $u \in L^2(\mathcal{G})$ of the differential system

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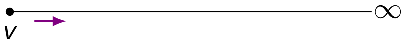
Here, Z is a set of degree-one vertices where we impose the homogenous Dirichlet boundary condition. For $\mathrm{v} \in \mathbb{V} \setminus Z$, the condition on the sum of derivatives is called *Kirchhoff's condition*.

The nonlinear Schrödinger equation

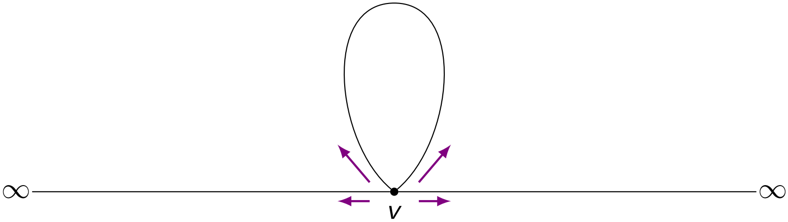
Kirchhoff's condition: degree-one nodes



$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(v+t)-u(v)}{t} = 0$$

The nonlinear Schrödinger equation

Kirchhoff's condition in general: outgoing derivatives



$$\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0$$

Variational formulations

Solutions of our problem correspond to critical points of the action functional J_λ defined by

$$J_\lambda(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\mathcal{G}} |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p \,\mathrm{d}x$$

on the Sobolev space

$$H^1_Z := \left\{ u : \mathcal{G} \to \mathbb{R} \;\middle|\; u \text{ is continuous; } u, u' \in L^2(\mathcal{G});\ \forall v \in Z, u(v) = 0 \right\}.$$

Goals

We are interested in the qualitative properties of

1) solutions minimizing the action on the set of nonzero solutions $\rightarrow$

the ground states (GS)

2) solutions minimizing the action on the set of nodal solutions $\rightarrow$

the nodal ground states (NGS)

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1) The GS is positive on $\mathcal{G}$
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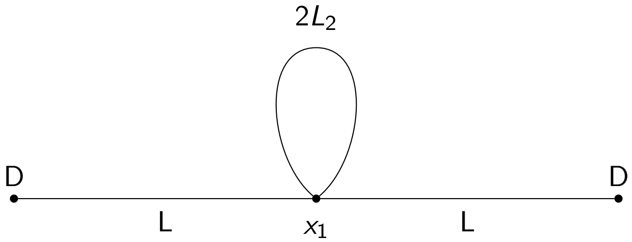
How do symmetries of the graph $\mathcal{G}$ transfer to symmetries of the GS and the NGS?

Where are the roots of the NGS? The maximum value? ...

Example 1 – The segment with two points glued together



What is the shape of the GS? The NGS?

The asymptotic regime $p \to 2$

Hope: obtain more information in the regime $p \approx 2$, by studying the *spectral* properties of the problem.

For every positive integer k and $p > 2$, we want to relate the solutions of the nonlinear problem

$$\begin{cases} -\tilde{u}'' + \lambda \tilde{u} = |\tilde{u}|^{p-2}\tilde{u} & \text{on every edge of } \mathcal{G}, \\ \tilde{u} \text{ is continuous} & \text{on } \mathcal{G}, \\ \sum_{e \succ \mathrm{v}} \tilde{u}_e'(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in \mathbb{V} \setminus Z, \\ \tilde{u}(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in Z, \end{cases}$$

to the eigenfunctions of the corresponding *eigenvalue problem* with eigenvalue γ_k.

A rescaling

In order to better understand the behaviour of the solutions as $p \to 2$, we consider the new variable $u = \gamma_k^{-1/(p-2)}\tilde{u}$. They are solutions of the nonlinear problem

$$\begin{cases} -u'' + \lambda u = \gamma_k |u|^{p-2} u & \text{on every edge of } \mathcal{G}, \\ u \text{ is continuous} & \text{on } \mathcal{G}, \\ \sum_{e \succ \mathrm{v}} u_e'(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in \mathbb{V} \setminus Z, \\ u(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in Z. \end{cases} \qquad (\mathcal{P}_{p,k})$$

The reduced problem when $p \approx 2$

Let $(u_{p_n})_n$ be a sequence of solutions to $(\mathcal{P}_{p_n,k})$, $(p_n)_n \subseteq]2, +\infty[$, $p_n \to 2$.

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Question

What can we say about u_*?

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Let $\varphi \in H^1_Z(\mathcal{G})$. Using φ as a test function in $(\mathcal{P}_{p_n,k})$, we get

$$\int_{\mathcal{G}} (u'_{p_n}\varphi' + \lambda u_{p_n}\varphi)\,\mathrm{d}x = \lambda_k \int_{\mathcal{G}} |u_{p_n}|^{p_n-2} u_{p_n}\varphi\,\mathrm{d}x.$$

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Question

Is that all we can say about u_*?

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Let us use specifically $\psi \in E_k$ as a test function in $(\mathcal{P}_{p_n,k})$. We obtain

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Thus,

$$\int_{\mathcal{G}} (|u_{p_n}|^{p_n - 2} - 1) \, u_{p_n} \psi \, \mathrm{d}x = 0.$$

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We divide by $p_n - 2$:

$$\int_{\mathcal{G}} \frac{|u_{p_n}|^{p_n-2} - 1}{p_n - 2} u_{p_n} \psi \, \mathrm{d}x = \int_{\mathcal{G}} \frac{\mathrm{e}^{(p_n-2)\ln|u_{p_n}|} - 1}{p_n - 2} u_{p_n} \psi \, \mathrm{d}x = 0.$$

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$$\int_{\mathcal{G}} (u_* \ln|u_*|)\psi \,\mathrm{d}x = 0.$$

Definition

A function $u_* \in E_k$ is a **solution of the reduced problem on** E_k if and only if

$$\int_{\mathcal{G}} (u_* \ln|u_*|)\psi \,\mathrm{d}x = 0$$

for all $\psi \in E_k$.

Recap

Given a sequence $(u_{p_n})_n$, $p_n \to 2$ converging weakly to $u_* \in H^1_Z$, we have seen that necessarily:

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Question

Given a solution of the reduced problem $u_* \in E_k$, can one find solutions of $(\mathcal{P}_{p,k})$ close to u_* for $p \approx 2$? Can one detect when there is only one solution close to u_* for a given $p \approx 2$?

Lyapunov-Schmidt reduction

Functional space with extra regularity:

$$H := \left\{ u \in H^1_Z \mid u \text{ is } H^2 \text{ in each edge}, u \text{ satisfies Kirchhoff's conditions} \right\}.$$

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We fix $k \geq 1$ and we define the map

$$F : \begin{cases} [2, +\infty[\times H & \to L^2(\mathcal{G}), \\ (p, u) & \mapsto -u'' + \lambda u - \lambda_k |u|^{p-2} u. \end{cases}$$

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$$F(p, u) = 0 \iff u \text{ solves } (\mathcal{P}_{p,k}).$$

Rough idea

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We will obtain good invertibility properties on $E_k^\perp$ and we are then reduced to a finite dimensional problem on E_k.

Rough idea - II

For $p \approx 2$, the leading term will actually be

$$\begin{aligned}\partial_p F(2,u) &= \partial_p\Big(-u'' + \lambda u - \lambda_k |u|^{p-2} u\Big)_{|p=2}\\ &= -\lambda_k u \partial_p \Big(e^{(p-2)\ln|u|}\Big)_{|p=2}\\ &= -\lambda_k u \ln|u|.\end{aligned}$$

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As we will see later, questions of "non-degeneracy" (in u) will thus involve

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hence expressions of the form

$$1 + \ln|u|.$$

A word of caution

Be careful!

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To perform the Lyapunov-Schmidt reduction around u_*, we will need

$$F : \begin{cases} [2, +\infty[\times H & \to L^2(\mathcal{G}), \\ (p, u) & \mapsto -u'' + \lambda u - \lambda_k |u|^{p-2} u. \end{cases}$$

to be $\mathcal{C}^2$ in u in the neighborhood of $(2, u_*)$.

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Remark: if $u \in E_k$, then

$$\left(u \in S \right) \iff u \text{ does not vanish identically on edges of } \mathcal{G}.$$

On graphs, this is **not** automatic: no unique continuation principle!

Nondegenerate solutions of the reduced problem

Definition

A solution $u_* \in E_k \cap S$ of the reduced problem on E_k is **nondegenerate** if and only if the map

$$E_k \to E_k : \psi \mapsto P_{E_k}\Big((1 + \ln|u_*|)\psi\Big)$$

is invertible.

Main Theorem

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Let $k \geq 1$ *be an integer and let* $u_* \in E_k \cap S$*.*

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1. ***non-existence***: *If u_* is not a solution of the reduced problem, then there exists a neighbourhood U of $(2, u_*)$ in $[2, +\infty[\times H$ so that problem $(\mathcal{P}_{p,k})$ has no solution in U with $p > 2$;*

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1. ***non-existence**: If u_* is not a solution of the reduced problem, then there exists a neighbourhood U of $(2, u_*)$ in $[2, +\infty[\times H$ so that problem $(\mathcal{P}_{p,k})$ has no solution in U with $p > 2$;*
2. ***existence, uniqueness and non-degeneracy**: If u_* is a nondegenerate solution of the reduced problem, then there exists a neighbourhood U of $(2, u_*)$ in $[2, +\infty[\times H$ and a number $\varepsilon > 0$ so that for all $p \in]2, 2 + \varepsilon]$, there exists a **unique** $u_p \in H$ so that (p, u_p) belongs to U and so that u_p is a solution of problem $(\mathcal{P}_{p,k})$.*

Unidimensional eigenspaces

In case $E_k = \operatorname{span}\varphi$ is of dimension 1, up to sign, we know exactly the limit u^* as we know that $u_* = a\varphi$ with a such that

$$0 = \int_{\mathcal{G}} \varphi^2 \ln|a\varphi| \,\mathrm{d}x = \int_{\mathcal{G}} \varphi^2 (\ln|a| + \ln|\varphi|) \,\mathrm{d}x.$$

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Moreover, it is nondegenerate.

Unidimensional eigenspaces - nondegeneracy

Saying that u_* is a solution of the reduced problem means that

$$P_{E_k}\left(u_* \ln |u_*|\right) = 0,$$

namely that

$$\int_{\mathcal{G}} |u_*|^2 \ln |u_*| \,\mathrm{d}x = 0.$$

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is invertible, namely that

$$\int_{\mathcal{G}} |u_*|^2 (1+\ln|u_*|) \, \mathrm{d}x$$

is nonzero.

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is invertible, namely that

$$\int_{\mathcal{G}} |u_*|^2(1+\ln|u_*|)\,\mathrm{d}x$$

is nonzero. The conclusion follows.

Uniqueness of positive solutions for $p \approx 2$

Theorem

If $p \approx 2$ is close enough to 2, the positive solution of $(\mathcal{P}_{p,1})$ is unique and is a ground state of the problem.

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Main ingredients of the proof.

- Show that there exists $C > 0$ such that all positive solutions of $(\mathcal{P}_{p,1})$ with $2 < p \leq 3$ satisfy $\|u\|_{H^1(\mathcal{G})} \leq C$;

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- When $p \to 2$, sequences of positive solutions to $(\mathcal{P}_{p,1})$ converge weakly (up to subsequences) to the only positive eigenfunction;
- Since $\dim E_1 = 1$, u_* is nondegenerate;
- The Lyapunov-Schmidt reduction proves the uniqueness result. □

Convergence of nodal ground states when $p \to 2$

Theorem (Convergence of nodal ground states)

If $(u_{p_n})_n$ is a sequence of nodal ground states of $(\mathcal{P}_{p,k})$ with $p_n \to 2$, then up to a subsequence one has that

$$u_{p_n} \xrightarrow[n\to\infty]{H^2} u_*,$$

where $u_* \in E_2 \setminus \{0\}$ is a solution of the reduced problem.

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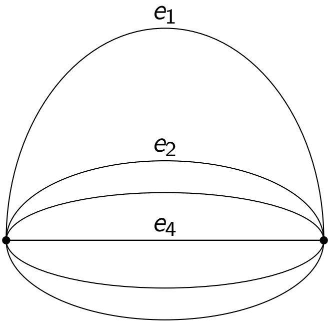
Remark

If u_* belongs to S (i.e. does not vanish on any edge) and is nondegenerate, one may obtain uniqueness and symmetry results by the Lyapunov-Schmidt reduction.

The n-bridge



n-edges e_1, ... e_n of length $2L_1$, ..., $2L_n$ with $L_1 > L_2 \geq L_3 \geq ... \geq L_n$.

What can be said on the ground state and the nodal ground state for $p \approx 2$?

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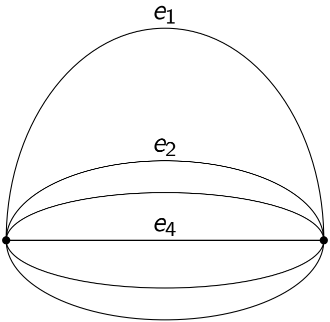

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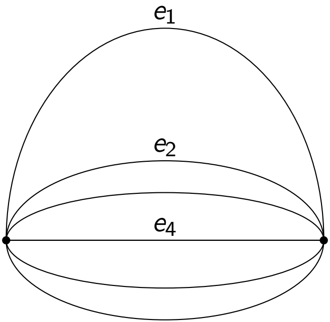

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What can be said on the ground state and the nodal ground state for $p \approx 2$?

Ground state: easy ... It is constant for $p \approx 2$. What about the nodal ground state?

The n-bridge

The second eigenspace

Let us parametrize the edges on $[-L_i, L_i]$. The solution of

$$\begin{cases} -u'' = \gamma u & \text{on } [-L_i, L_i], \\ u \text{ is continuous} & \text{on } \mathcal{G}, \\ \sum_{e \succ \mathrm{v}} \tilde{u}_e'(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in \mathbb{V} \end{cases}$$

are given by $u_i(x) = a_i \cos(\sqrt{\gamma} x) + b_i \sin(\sqrt{\gamma} x)$ with, for all $1 \leq i, j \leq n$,

$$\begin{cases} a_i \cos(\sqrt{\gamma} L_i) = a_j \cos(\sqrt{\gamma} L_j), \\ b_i \sin(\sqrt{\gamma} L_i) = b_j \sin(\sqrt{\gamma} L_j), \\ \sum a_i \sin(\sqrt{\gamma} L_i) = 0, \\ \sum b_i \cos(\sqrt{\gamma} L_i) = 0. \end{cases}$$

The n-bridge
The second eigenspace

We prove that the second eigenvalue is defined by

$$\gamma_2 \in \left] \left(\frac{\pi}{2L_1}\right)^2, \ \min\left\{\left(\frac{\pi}{2L_2}\right)^2, \left(\frac{\pi}{L_1}\right)^2\right\}\right[\text{ is the solution of}$$

$$\sum \tan(\sqrt{\gamma_2} L_i) = 0,$$

with eigenfunction

$$\begin{cases} \varphi_{2,1}(x) &= a_1 \cos(\sqrt{\gamma_2} x), \\ \varphi_{2,i}(x) &= a_1 \dfrac{\cos(\sqrt{\gamma_2} L_1)}{\cos(\sqrt{\gamma_2} L_i)} \cos(\sqrt{\gamma_2} x), \quad \text{for } i \geq 2 \end{cases}$$

The n-bridge

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5. if $L_i = L_j$ then $\varphi_{2,i}(x) = \varphi_{2,j}(x)$.

The n-bridge
Conclusions

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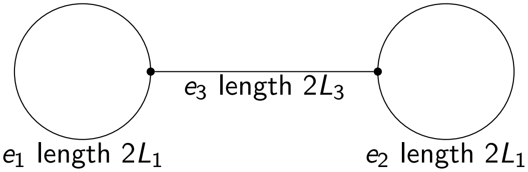
The dumbbell



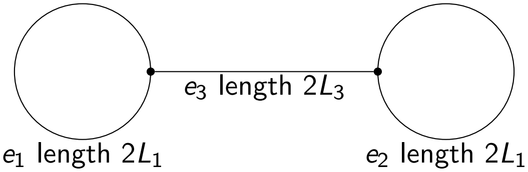
$$\gamma_2 \in \left]0, \left(\frac{\pi}{2(L_1+L_3)}\right)^2\right[\text{ sol. of}$$

$$\cos(\sqrt{\gamma_2}(L_3 - L_1)) - 3\cos(\sqrt{\gamma_2}(L_1 + L_3)) = 0,$$

$$\text{with eigenfunction} \begin{cases} \varphi_{2,1}(x) & = & a_1 \cos(\sqrt{\gamma_2}x), \\ \varphi_{2,2}(x) & = & -a_1 \cos(\sqrt{\gamma_2}x), \\ \varphi_{2,3}(x) & = & a_3 \sin(\sqrt{\gamma_2}x), \end{cases}$$

where $-a_1 \cos(\sqrt{\gamma_2}L_1) = a_3 \sin(\sqrt{\gamma_2}L_3)$ so that

The dumbbell



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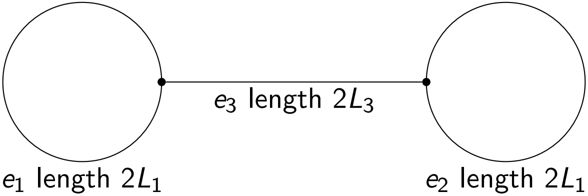
$$\text{where } -a_1\cos(\sqrt{\gamma_2}L_1) = a_3 \sin(\sqrt{\gamma_2}L_3) \text{ so that}$$

1. $\varphi_{2,1}$, $\varphi_{2,2}$ are even;
2. the root of φ_2 is the middle point of e_3;

The dumbbell – Conclusion

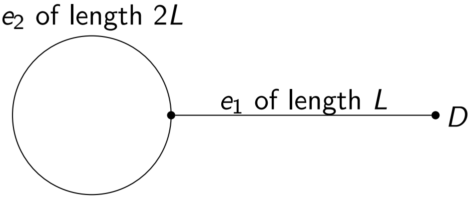


Ground state: It is constant for $p \approx 2$.

Nodal ground state: For $p \approx 2$

1. u_1, u_2 are even,
2. the root of u is the middle point of e_3,
3. u is odd “globally”.
4. u_3 is strictly monotone.
5. the maximum of φ_2 is the “extremity” of the loop.

The tadpole – Dirichlet

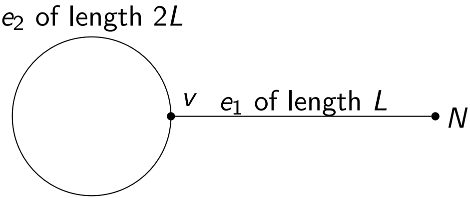


For $p \approx 2$, the positive GS is even on the loop, increasing on the segment.

For $p \approx 2$, the NGS is

1. even on the loop;
2. one nodal domain is included in the loop;
3. the maximum of the amplitude is in the interior of the segment.

The tadpole – Neumann



For $p \approx 2$, the GS is constant

The second eigenvalue is $\gamma_2 = (\frac{\pi}{2L})^2$ with eigenfunction

$$\varphi_{2,1}(x) = -2a_2 \cos\left(\frac{\pi}{2L}x\right), \quad \varphi_{2,2}(x) = a_2 \cos\left(\frac{\pi}{2L}x\right)$$

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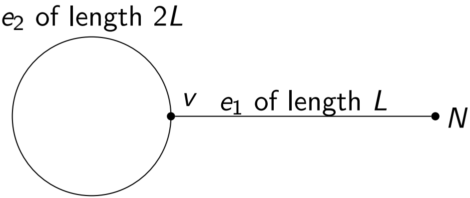

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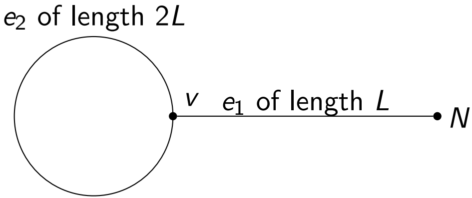
1. φ_2 is even on the loop;
2. the loop is one nodal domain, the segment is the second nodal domain, the nodal set is the vertex v;
3. the maximum of the amplitude is on the vertex of degree 1.

The tadpole – Neumann

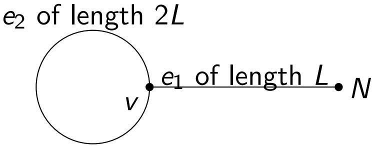


What about the NGS for $p \approx 2$?

Easy:

1. u is even on the loop;
2. the maximum of the amplitude is on the line.

The tadpole - Neumann



What about the nodal domain? $u(v) = 0$ or not?

The tadpole - Neumann

u cannot be equal to zero at the vertex v as otherwise u_2 is a solution of

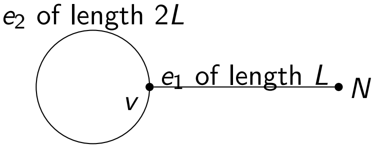
$$\begin{cases} -u'' + \lambda u = |u|^{p-2} u \\ u(-L) = u(L) = 0 \\ u > 0 \text{ on }]-L, L[\end{cases}$$

and u_1 is a solution of

$$\begin{cases} -u'' + \lambda u = |u|^{p-2} u \\ u'(0) = u(L) = 0 \\ u < 0 \text{ on }]0, L[\end{cases}$$

By uniqueness of the solution of these problems and as $u_1 = -u_2|_{[0,L]}$, this is not a solution on the graph as it does not satisfy the Kirchhoff condition.

The tadpole – Neumann



In fact $u^{-1}(0) = \{x_0\}$ with x_0 a point of the segment.

$$\varphi_{2,1}(x) = -2a_2 \cos(\frac{\pi}{2L}x), \quad \varphi_{2,2}(x) = a_2 \cos(\frac{\pi}{2L}x)$$

hence the amplitude is larger on the segment.

The same is true for the NGS by convergence.

The tadpole – Neumann

We know that the time needed to go from the maximum to 0 is a decreasing function of the value of the maximum. Hence the result.

The tadpole – Neumann

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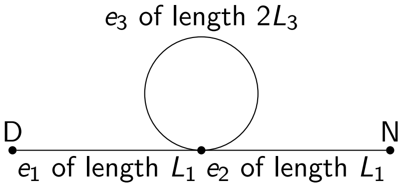
Conclusion: For $p \approx 2$, the NGS satisfies :

1. u is even on the loop;
2. the maximum of the amplitude is on the line;
3. $u^{-1}(0) = \{x_0\}$ with x_0 a point of the segment;
4. one nodal domain is included in the segment, the other contains the loop.

One bubble



First eigenvalue: $\gamma_1 = \left(\frac{\pi}{2(L_3+2L_1)}\right)^2$ with the first eigenfunction even on the loop. The GS is even on the loop.

One bubble

Second eigenvalue:

If $L_3 < 4L_1$ then $\gamma_2 = \left(\frac{3\pi}{2(L_3 + 2L_1)}\right)^2$ is simple with the second eigenfunction even on the loop and not identically zero on an edge. In that case, the NGS is also even on the loop.

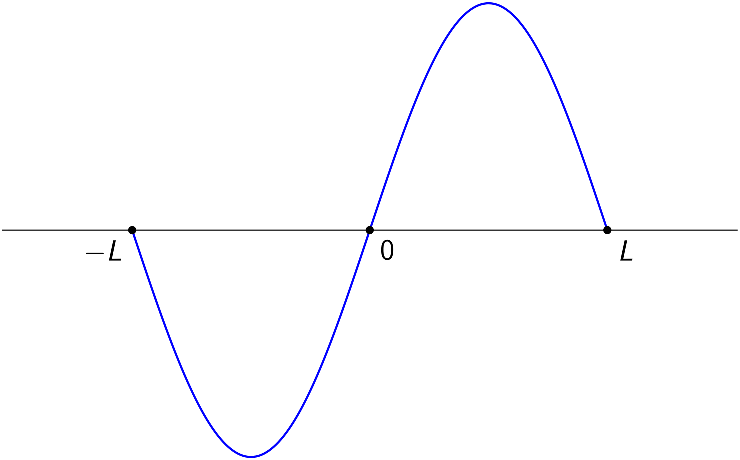
One bubble

Second eigenvalue:

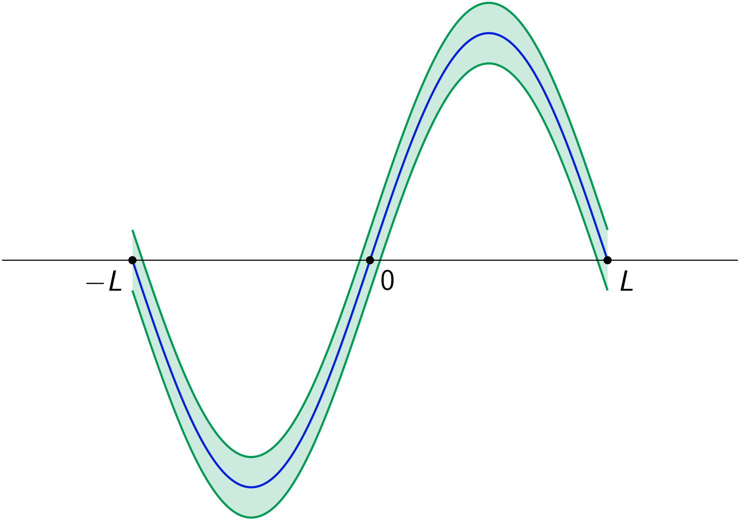
If $L_3 < 4L_1$ then $\gamma_2 = \left(\frac{3\pi}{2(L_3 + 2L_1)}\right)^2$ is simple with the second eigenfunction even on the loop and not identically zero on an edge. In that case, the NGS is also even on the loop.

If $L_3 > 4L_1$ then $\gamma_2 = \left(\frac{\pi}{L_3}\right)^2$ is simple with the second eigenfunction odd on the loop and identically zero on e_1 and on e_2. What about the NGS?

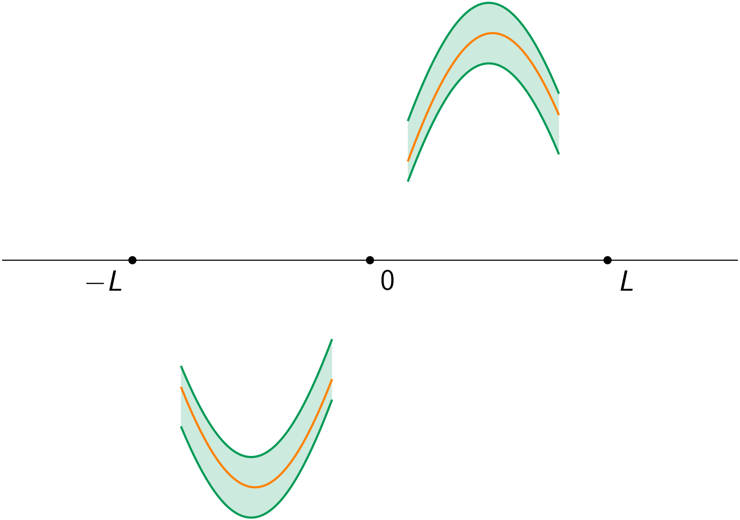
The limit when $p \to 2$ on the loop



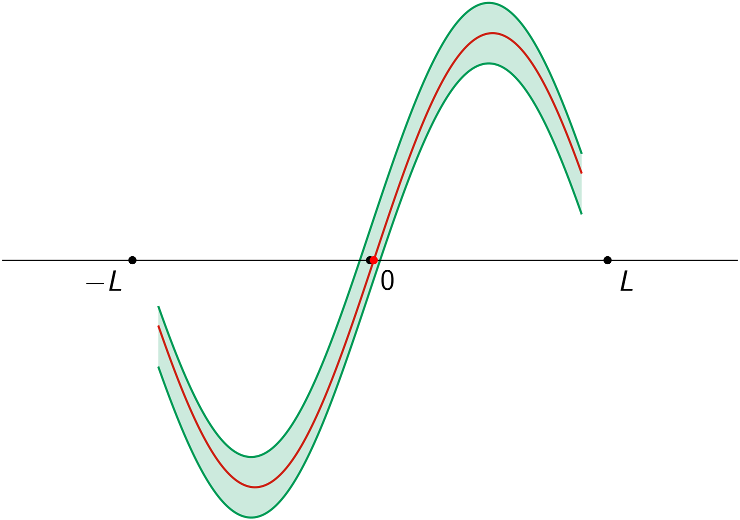
Box when $p \approx 2$ on the loop



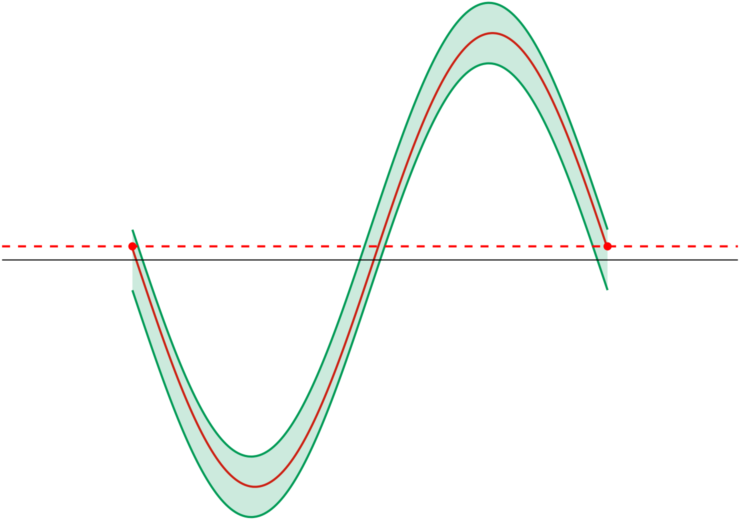
Sign change on the loop



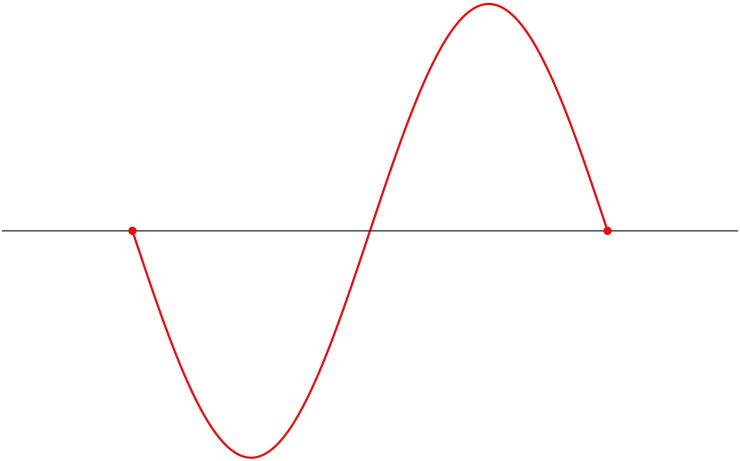
Problem: behaviour at the node?



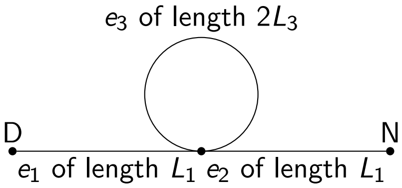
Continuity



Conclusion: On the loop



One bubble – Asymmetric vertex conditions

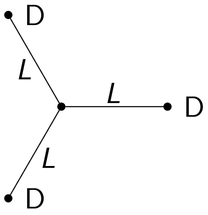


The GS is even on the loop.

If $L_3 < 4L_1$ then the NGS is also even on the loop.

If $L_3 > 4L_1$ then the NGS is odd on the loop and identically zero on e_1 and on e_2.

The symmetric stargraph – Symmetry breaking



For L fixed, by uniqueness, for $p \approx 2$, the GS is *symmetric* (i.e. its restrictions to all edges, viewed as functions $[0, L] \to \mathbb{R}$, are all equal).

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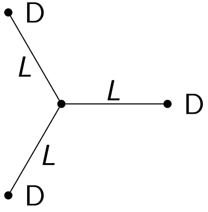

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Instead, for any $p > 2$, if L is large enough then the ground state on $\mathcal{G}_L$ is *not* symmetric.

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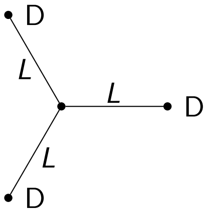

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Instead, for any $p > 2$, if L is large enough then the ground state on $\mathcal{G}_L$ is *not* symmetric.

In particular, the uniqueness of the positive solution is not always valid (not as on the interval).

Thanks for your attention!

References

NLS on metric graphs

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